Life-cycle performance of structures subject to multiple deterioration mechanisms

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ABSTRACT

This paper studies structural deterioration as a result of the combined action of progressive degradation (e.g., corrosion, fatigue) and sudden events (e.g., earthquakes). The structural condition at a given time is measured in terms of the system’s remaining life, which is defined in practice by an appropriate structural performance indicator (e.g., inter-story drift). Structural reliability is evaluated against prescribed design and operation thresholds that can be used to establish limit states or intervention policies. It is assumed that sudden events conform to a compound point process with shock sizes and interarrival times that are independent and identically distributed random variables. Progressive deterioration is initially modeled as a deterministic function. Randomness is later included also as a shock process with times between random deterioration jumps described by a suitable deterministic function. Structural performance with time is modeled as a regenerative process and an expression for the limiting average performance is obtained. The model is illustrated with some examples and compared with similar models showing the importance of including the damage history when studying the life-cycle performance of infrastructure systems.

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1. Introduction

The maintenance and operation of large infrastructure systems represents a significant investment in both the public and private domains. As in any engineered system, different geographical and environmental conditions in which these systems operate can produce widely varying rates of deterioration over time. To mitigate the risks involved with infrastructure deterioration, including loss of performance and potential public safety hazards, requires a commitment of financial resources to repair, replacement, and redesign activities. Analytical and computational modeling of deterioration is crucial to the wise use of these resources. The deterioration of large structural systems represent trillions of dollars of public and private investment in the US alone, and many of these systems are facing major repair or complete replacement decisions in the coming decades.

In practice, most infrastructure systems deteriorate as a result of the action of both sudden extreme events (i.e., shocks) and continuous (progressive) degradation caused mainly by aging and environmental factors. Earthquakes and hurricanes are examples of extreme events, while corrosion and fatigue are typical cases of progressive degradation. Both mechanisms cause damage to accumulate with time, effectively depleting the remaining life of the system. In civil engineering infrastructure systems, remaining life is described by a system performance measure usually evaluated in terms of structural capacity (e.g., material resistance, displacement – drift).

Structural deterioration (damage) can be managed with a variety of intervention measures, which can be grouped into preventive maintenance (i.e., activities designed to increase the remaining life of the system by repairing or replacing aged components), essential maintenance (i.e., maintenance necessary to avoid imminent failure), and full reconstruction [22,29,30,39]. The need for an intervention is usually defined by performance thresholds, which are prescribed in design codes or operation manuals. The main goal in infrastructure management is to maximize the system availability at a minimum cost; that is, maintaining the system operating in acceptable conditions during the maximum length of time. This objective is achieved by balancing the economic investment, the benefits derived from the existence of the project and the consequences in case of failure. In other words, designing, constructing and maintaining infrastructure may be viewed as a decision problem where the maximum economic benefit of the life-cycle of the project is achieved while the reliability requirements are fulfilled simultaneously at the decision point [33].

Structural damage accumulation with time is a topic that has been widely discussed and studied through the so-called life-cycle analysis; however, only few analytical solutions have been
2. Structural deterioration

Structural deterioration is concerned with a decay of one or more structural properties (e.g., stiffness, resistance); therefore, there are many ways in which structural deterioration can be characterized. In most engineering systems the deterioration process is divided in progressive and sudden deterioration. The former describes a slow process caused mainly by environmental factors, while the latter describes sudden changes in the structural capacity.

Progressive deterioration in reinforced concrete structures is a reduction of the structural capacity caused mainly by chloride ingress, which leads usually to steel corrosion, loss of effective cross-section of steel reinforcement, concrete cracking, loss of bond and spalling. The details of these processes are beyond the scope of the paper but are well described by, for instance, Val and Stewart [42] and Liu and Weyers [25]. Progressive deterioration has been evaluated in the past by using the reliability index profile, a function that describes the change of the reliability index with time. Simplified degradation models have been proposed by Ellingwood and Mori [11], Mori and Ellingwood [27], Frangopol et al. [15], Petcherdchoo et al. [32], Cinlar et al. [7], and Dawson [9]. Many papers have been published proposing models to keep track of structural deterioration with time and to define optimal intervention policies. Commonly, these models are based on Markov Decision Processes (MDP) [18,16,22,26,17,23]. Other methods that have been used include Bayesian probability [31,21], renewal theory [40] and approximate functions obtained from experimental data [27]. A review of common probabilistic models for life-cycle performance of deteriorating structures can be found in Frangopol et al. [15].

Deterioration caused by extreme events is mostly associated with earthquakes but can be used also to model the effect of hurricanes or blasts (terrorists attacks). In these cases, the structure is subject to randomly occurring shocks, and each shock causes a random amount of damage. Most shock maintenance and failure models are based on a control-limit policy, in which an intervention is carried out once the accumulated damage exceeds a critical value or at failure, whichever occurs first [43]. Extensive research has been carried out on mathematical models for shock degradation [23,9,1,41,28,30,12,14,47]. In addition, the effect of both shocks and progressive deterioration has also been studied by Wortman et al. [45] and Yang and Klutke [46]. Although this problem has been discussed extensively in civil engineering related problems, few analytical solutions have been proposed within the context of structural optimization and life-cycle cost analysis. The first works on this topic were published by Rosenblueth and Mendoza [35], Hasofer [19] and Rosenblueth [36] in the context of earthquake-resistant design optimization. Their ideas were reconsidered by Rackwitz [33] to propose a general framework for optimal design and reliability verification. Rackwitz considered two possible structural operation policies: (1) structures abandoned after first failure; and (2) structures reconstructed systematically. Furthermore, for each case three loading conditions were considered: (1) time invariant loads; (2) extreme loads (performance under Poisson perturbances); and (3) failures described by Poisson perturbances. Rackwitz uses renewal theory and takes advantage of the form of the discount function to achieve a rather practical (time-independent) solution to the expected life-cycle cost of a structure. The merit of the solution for random failures in time with systematic reconstruction (e.g., seismic design case) is that it does not depend on the specific lifetime of the structure, which is a random variable very difficult to quantify and usually underestimated by codes of practice. The solution is based on failure intensities and not on time-dependent failure probabilities. It is neither necessary to define arbitrary reference times of intended use nor it is necessary to compute first-passage time distributions. A particular application of this model to earthquake damage of structures can be found in Sanchez-Silva and Rackwitz [34,38]. Similarly, and also within the context of extreme loads (i.e., earthquakes and winds), Wen and Kang [44] developed a model to minimize the life-cycle cost with respect to the design load and resistance. The random occurrence and intensity variation of the hazards in time is described by a simple random process. The model also includes several possible damage states after an event with the corresponding probability of reaching every damage state. Several commercial reliability analysis software packages have been developed that manage the combined problem of deterioration and extreme events. In particular, it is important to mention COMREL [8] (i.e., COMREL-TV) that considers time-variant reliability by the out crossing approach for stationary or non-stationary cases using regular or intermittent rectangular wave processes and differentiable Gaussian and non-gaussian translation processes (Hermite or Nataf processes) [8].

Thus, based on the previous discussion, the model presented in this paper takes into consideration the following aspects:

1. Structural deterioration is a combination of both progressive and sudden damaging events.
2. The deterioration history plays an important role in estimating the structural condition at a given time.
3. Multiple damage states (not only failure and non-failure states) should be taken into account.
4. After failure, structural reconstruction does not necessarily take the component to its initial condition.
5. The selection of acceptable operational and failure thresholds is critical in life-cycle analysis.
3. General deterioration model

3.1. Description of damage accumulation

General life-cycle models describe the performance (i.e., deterioration) of a system or a component throughout its lifetime. The deterioration of the system is commonly measured with respect to a physical parameter; for instance, in structures it can be the stiffness or the inter-story drift. In order to be general, the parameter that will be used to describe the system performance in this paper is the structural capacity. In a life-cycle model, once a structure is put in service damage starts accumulating as a result of progressive deterioration or sudden events (i.e., shocks) until the structure fails; it is then repaired or reconstructed and the process restarts. A sample path describing the performance of structural system throughout its lifetime is depicted in Fig. 1.

Structural deterioration (i.e., damage) is defined as any change to the material or the geometric properties affecting the structural capacity. Consider a structural component with an initial capacity $u_0$ (Fig. 1). Then, if $D(t)$ describes the accumulated deterioration of the component at time $t$ (in capacity units); before the first failure, the capacity of the component by time $t$ can be expressed as:

$$V(t) = u_0 - D(t)$$

Furthermore, based on the assumption that the structure is subject to both continuous and sudden damaging events and that they are independent, the deterioration by time $t$ can be computed as:

$$D(t) = \int_0^t r_p(\tau)d\tau + \sum_{i=1}^{N_t} Y_i$$

where $N_t$ is the number of shocks by time $t$, $Y_i$ is the loss of capacity caused by shock $i$ (i.e., sudden event type) and $r_p(t) > 0$ describes the rate of some continuous progressive deterioration process. Then, combining Eqs. (1) and (2), the remaining capacity of the structure by time $t$ becomes,

$$V(t) = u_0 - \int_0^t r_p(\tau)d\tau + \sum_{i=1}^{N_t} Y_i$$

Once the structural capacity falls below a given performance threshold (e.g., $k^*$ or $s^*$ in Fig. 1), the structure is intervened/reconstructed, leading to a new initial capacity $u_{i+1}$, where the sub-index $i$ indicates the cycle in which the system is at. In Section 4 we consider the situation where damage is caused solely by shocks; i.e., the middle term in Eq. (3) is zero. Then, in Section 5, we include progressive deterioration in the model.

3.2. Intervention criteria

An intervention (maintenance or reconstruction) is carried out once the capacity of the system reaches a threshold value (i.e., limit state). Two possible threshold values are identified here; i.e., $s^*$ or $k^*$ (Fig. 1). The first one corresponds to the minimum structural performance level; this can be thought of as a level below which the system cannot be in service under any circumstance; i.e., $s^*$ in Fig. 1. The selection of $s^*$ is not usually specified in codes of practice but obtained based on experience. For structures it may be defined, for example, as the inter-story drift that makes the structure extremely flexible (e.g. $\Delta \geq 3–5\%$) or as a certain level of vibrations on a bridge. In most structures, the threshold $s^*$ corresponds to the ultimate structural capacity and therefore it is equivalent to structural collapse.

The second limiting value is an operating threshold; i.e., $k^*$ in Fig. 1. This value describes the minimum acceptable operation level defined, for instance, by a regulatory agency. Although the system can still operate below this level, the operation will not be considered satisfactory, efficient or acceptably safe. This limit indicates the need for a preventive intervention or structural maintenance. Maintenance programs are based on early repairs; that is, preventive repairs carried out before the structure collapses. Note that failure and preventive maintenance are mutually exclusive since for any instantaneous event that takes the accumulated damage below the threshold $k^*$, only one of these two cases is possible.

Based on a model whose performance is defined by the two limits stated above, we considered the following possible decisions:

1. if the capacity level is below threshold $s^*$, the structure is fully replaced;
2. if the capacity is between $k^*$ and $s^*$, preventive maintenance is carried out; and
3. if the capacity is larger than $k^*$ the structure is not modified (Fig. 1).

3.3. Cyclic structural performance

Throughout the structure’s lifetime it is expected that the accumulated damage causes the capacity to fall below one of the operational limits described in the previous section. Then, if the structure is not abandoned after failure but systematically intervened (e.g., reconstructed), it is said that the system regenerates. This means that there are times (i.e., cycles) at which the system restarts itself.

During one cycle, it is said that the structure will be in acceptable operation (i.e., “on”) as long as the capacity is larger than a given value $k^*$ (Fig. 1). On the other hand, the system will be out of service (i.e., “off”) the length of time during which the capacity is below $k^*$; and until it is intervened (e.g., reconstructed/repaired/retrofitted) and taken to a new “initial” capacity $u > k^*$. In the case of infrastructure systems, intervention and replacement times are usually small compared with the total life of the structure; therefore, they can be assumed to be “instantaneous”. In many models the downtime is important and, therefore, the focus of the analysis is on describing the system availability. However, by assuming an “instantaneous” repair model, our paper concentrates on modeling the variation of capacity with time instead of availability.

Based on this assumption, the cyclic process of operation, failure and immediate repair can be modeled as a counting process.
In what follows we will assume conditions under which the times between interventions are independent and identically distributed (iid), with an arbitrary distribution; such a process is called a renewal process [37]. The renewal process assumption requires that the intervention criteria (i.e., \( k \) or \( s' \)) is kept constant; and only one intervention (maintenance/replacement) criteria can be used during the entire structure’s lifetime analysis.

4. Structures subject to extreme overloads (i.e., shocks degradation)

This section presents a model that describes the deterioration of structures subject to successive shocks only. The proposed model is based on the following assumptions:

1. damage accumulates as a consequence of consecutive shocks;
2. the sizes of shocks (damage) are independent and identically distributed;
3. repairs and/or replacements are instantaneous; therefore, the system does not remain “off” (i.e., out of service) at any time;
4. a single threshold \( \alpha \) is used as the criteria for intervention; and
5. all replacements and repairs return the system to a condition as good-as-new – i.e., we consider statistically identical cycles.

In this section, we study the structural performance of the system in terms of the instantaneous failure probability. In so doing, our aim is to characterize the dependence of failure probability on the probabilistic capacity of a new or repaired structure, the probabilistic description of the shock damage process, and the threshold decision variable \( \alpha \).

4.1. Formulation of the stochastic performance of the system

4.1.1. Occurrence of shocks in time

Extreme events that caused sudden damage such as earthquakes or hurricanes are rare events that occur randomly in time and can be modeled using marked point process. Under our assumptions, interarrival times of successive events constitute a sequence of nonnegative independent random variables \( X_1, X_2, \ldots \), with common distribution \( F(t) = P(X < t) \), \( t = 1, 2, \ldots \) (Fig. 1) and density \( f(t) \).

We denote by \( \{N_t, t \geq 0\} \) the associated renewal counting process of arriving shocks. For the renewal counting process, the intensity or hazard rate \( \lambda(t) \) of the shock process can be expressed as:

\[
\dot{\lambda}(t) = \sum_{k \geq 1} \frac{f(t-T_k)}{E[T_k]} \cdot 1_{[T_k<T_{k+1}]},
\]

where \( T_k = \sum_{n=1}^{k} X_n \) is the time of the \( n \)th shock and \( 1_{[T_k<T_{k+1}]} \) is an indicator random variable of the set \( A \).

Informally, the hazard rate at time \( t \) gives the probability that a shock occurs in a small interval after time \( t \), given the history of the shock process prior to \( t \). Eq. (4) can be deconstructed as follows. The term \( 1_{[T_k<T_{k+1}]} \) is an indicator function indicating that the time \( t \) is evaluated within the cycle defined by \( [T_k,T_{k+1}] \). If the last shock occurred at time \( T_m \), the numerator can be seen as (roughly) the probability that the time between shocks is \( t - T_m \) and the denominator expresses the probability that no shock has yet occurred in the time interval \( [t - T_m] \).

4.1.2. Damage accumulation with time

Each shock carries with it an associated mark (i.e., damage amount) described by a random variable \( Y_i \), where the \( \{Y_i, i = 1,2,\ldots\} \) are positive, independent and identically distributed random variables with distribution \( G(y) = P(Y \leq y) \). The sequence of marks \( Y_i \) describes the shock magnitude; i.e., the amount of damage (in units of capacity) caused by the \( i \)th shock. Note that the magnitude of a shock does not describe the actual intensity of the event, but the damage caused on the structural system. The marked point process \( \{[T_i,Y_i], i = 1,2,\ldots\} \) is commonly called a renewal reward process. If no intervention takes place in the time interval \( [0,t] \), the accumulated damage at time \( t \) is given by \( \sum_{i=1}^{N_t} Y_i \), where \( N_t \) counts the total number of shocks by time \( t \).

The model assumes that the structure is returned to a condition as good-as-new each time the capacity falls below the threshold \( \alpha \). Therefore, the structural capacity can be described, over time, via successive independent and stochastically identical cycles. Then, if \( Z_i \) is defined as a random variable that denotes the time of the \( i \)th structural replacement (end of cycle \( i \)) with \( Z_i \geq 0 \), the deterioration up to time \( t \) can be computed as:

\[
Q(t) = \sum_{i=1}^{N_t} Y_i - \sum_{i=1}^{N_t} Y_i \cdot 1_{[Z_i < Z_{i+1}]} + \sum_{i=1}^{N_t} Y_i \cdot 1_{[Z_i < Z_{i+1}]}, \quad i = 2, 3, \ldots,
\]

where a sum is taken to be zero if it is empty, and \( 1_{[Z_i < Z_{i+1}]} \) is an indicator function that takes a value of 1 if \( Z_i < t < Z_{i+1} \) (i.e., \( t \) is in the \( i \)th cycle) and 0 otherwise. At the beginning of cycle \( i \), the capacity is reset to a random value \( u_i \). Therefore, the capacity at time \( t \) is computed by subtracting the accumulated damage from the total added capacity up to time \( t \); that is (Fig. 2).

\[
V(t) = \sum_{j=0}^{\infty} u_i \cdot 1_{[Z_i \leq t]} - Q(t)
\]

4.1.3. Formulation of intervention rate

At this point, it is assumed that interventions occur only at times of shocks. Then, for an intervention to take place at time \( t \), it is necessary that a shock occurs at \( t \) and that the size of the damage caused by the shock is large enough to cause the capacity to fall below the threshold \( \alpha \) (Fig. 3).

Define \( \{L_t, t \geq 0\} \) as the counting process of interventions; i.e., \( L_t \) is the number of interventions (i.e., either early or full replacement, depending on which threshold value is used) by time \( t \) with \( L_0 = 0 \). Since \( \{L_t, t \geq 0\} \) increases only by jumps of size one, we can think of the instantaneous rate of interventions as the function \( E[dL_t|L_t] \), where \( L_t \) denotes the history of interventions and shock processes. Since the occurrence times of shocks and shock sizes are assumed to be independent, we can write, in infinitesimal terms, \( dV(t) = E[dL_t|L_t] = P(dL_t = 1|L_t) = \dot{\lambda}(t) \int_{y>\alpha} dG(y) \).

\[
\dot{\lambda}(t) := E[dL_t|L_t] = P(dL_t = 1|L_t) = \dot{\lambda}(t) \int_{y>\alpha} dG(y) \quad (7)
\]

where \( V(t) = V(t) - \alpha^2 \); \( \dot{\lambda}(t) \) is the hazard rate defined in Eq. (4); and \( dG(y) = g(y)dy \).

The function \( V(t) \) can be interpreted as the instantaneous intervention (failure or maintenance) intensity (similar to a hazard rate or failure rate for random variables). It has the heuristic interpretation that, for small \( dt \),

\[
V(t)dt \approx P(\text{intervention in } [t, t + dt]|L_t)
\]

For the cases considered in this paper, the instantaneous intervention rate can be written as:

\[
V(t) = \begin{cases} \int_{y>\alpha} dG(y) \dot{\lambda}(t) & \text{Failure} \\ \int_{y\leq\alpha} dG(y) \dot{\lambda}(t) & \text{Prev. Maintenance} \end{cases}
\]

The limits of the integral in each case of Eq. (9) define the range of the shock size required to cause the intervention (failure or maintenance) (Fig. 3).

The distribution \( G \) describes the probability of having a certain damage level as a result of a shock (e.g., loss of structural capacity
after an earthquake). This distribution can be determined by considering the probability distributions of the demand imposed on the system and that of the structural response. Commonly $G$ is derived from the so-called fragility curves, which describe the probability that the system reaches a certain damage level in terms of a demand parameter. Several approaches to compute these curves have been proposed in the literature but their discussion is beyond the scope of this paper (see: [20,10]). Finally, it is important to stress that the model proposed in this paper assumes that the distribution $G$ does not depend upon the previous damage states of the system. Although this assumption might seem strong, it relaxes the assumption of several other models (e.g., [33]), which consider that the structural damage, as a result of successive shocks, does not accumulate with time.

4.1.4. Evaluation of the damage accumulated in a cycle

The complexity of the expression for $v(t)$ (Eq. (9)) lies in computing the limits of the integral, which depend on the number of shocks in the cycle and the probability distribution of the sum of shock sizes (i.e., accumulated damage). The limits of the integral can be defined, for any cycle, as (Fig. 3):

$$V(t, a^*) = V(t) - a^* = u_{t, -1} - a^* - Q(t)$$

(10)

Given that $Q(t)$ is equivalent to the sum of shocks (total accumulated damage) in cycle $L_t$ (Eq. (5)), Eq. (10) can be re-written as,

$$V(t, a^*) = u_{t, i - 1} - a^* \left[ \sum_{j=1}^{N_i} Y_j - \left\{ \sum_{j=1}^{N_{i-1}} Y_j \right\} \cdot 1_{a^*<Z_{i+1}} \right],$$

$$i = 2, 3, \ldots$$

(11)
In Eq. (11), $V(t, a')$ is a random variable that is function of the sum of shocks. Since the damage caused by a shock is distributed as $G(y) = P(Y \leq y)$, the sum on $n$ shocks will follow a distribution $G^{(n)}(y)$ that can be computed as the $n$-fold Stieltjes convolution of $G(y)$ with itself:

$$
\sum_{j=1}^{n} Y_j \sim G^{(n)}(y) \quad (12)
$$

then, by conditioning on the number of shocks:

$$
V(t, a', n) = \int_{a - a'}^{a - a'} (u_{i - 1} - a' - y)dG^{(n)}(y) \quad (13)
$$

Note that this expression is only valid as long as the accumulated damage is smaller than the total available capacity of the system, i.e., $G^{(n)}(y) \leq (u_{i - 1} - a')$ (Fig. 3); Otherwise, the system has already failed.

4.2. System abandoned after first intervention

This section considers the case of a structure exposed to successive shocks until it fails and then it is abandoned; this is, the system is not reconstructed or intervened in any way after failure. Under these assumptions, Eq. (9) specializes to:

$$
\mathcal{V}(t) = \int_{t_{s'}(t, s')/C_0}^{\infty} dG(y) \int_{t_{s'}(t, s')/C_0}^{\infty} \lambda(t) d\mathcal{P}(N = n) \quad (14)
$$

with $a' = s'$ or $a' = k'$ depending of the problem at hand. The lower limit of the integral is a function of the initial capacity and the shock history (Eq. [13]); then,

$$
V(t, a', n) = \int_{0}^{a - a'} (u_{i - 1} - a' - y)dG^{(n)}(y) \quad (15)
$$

In Eq. (15), the number of shocks occurring prior to failure is a random variable. Then, if the probability of having $n$ shocks by time $t$, $P(N = n, t)$ is known, the condition on $n$ can be removed and the instantaneous failure rate can be re-written as:

$$
\mathcal{V}(t) = \sum_{n=0}^{\infty} \left[ \int_{t_{s'}(t, s')/C_0}^{\infty} dG(y) \right] \int_{t_{s'}(t, s')/C_0}^{\infty} \lambda(t) d\mathcal{P}(N = n) \quad (16)
$$

where $P(N = n)$ indicates the probability that the expected number of shocks by time $t$. In addition, the probability that an intervention is required before or at time $t$ is obtained by integrating from 0 up to time $t$, i.e.,

$$
\Lambda(t) = \int_{0}^{t} \int_{0}^{\infty} \left[ \int_{t_{s'}(t, s')/C_0}^{\infty} dG(y) \right] \lambda(t) d\mathcal{P}(N = n) d\tau \quad (17)
$$

4.2.1. Illustrative example

Consider the case of a structure located in a region of high seismic activity in which the occurrence of earthquakes with magnitude $M > 4$ (those which can actually cause some damage) can be modeled as a Poisson process with rate $\mu = 1/year$. For the purpose of this example, the structural performance is described as a percentage of its initial capacity, which is distributed lognormally with $\mu = 100$ (%) and $COV = 0.25$. The initial capacity of the structure decays as earthquakes hit it and cause some damage with a threshold limit $s' = 0.25$. Assume further that the damage caused by an earthquake is governed by an exponential distribution $G(y, 0)$ with parameter (rate) $\theta = 0.05$. Under these assumptions, the two cases shown in Fig. 4 will be studied.

According to Eq. (4), the instantaneous occurrence rate of the earthquakes, with $T_0 = 0$ and $T_{n+1} = T_1$, is:

$$
\lambda(t) = \sum_{n=0}^{\infty} \frac{f(t - T_n)}{1 - \int_{0}^{\infty} f(x)dx} \mathbf{1}_{\{t < T_n + 1\}} = \frac{f(t)}{1 - \int_{0}^{\infty} f(x)dx} \mathbf{1}_{\{0 < T_1\}} = \frac{f(t)}{F(t)} \quad (18)
$$

where $f(t)$ is the probability density function of the time between earthquakes and $F(t)$ is the complementary distribution function: $F(t) = 1 - F(t)$. Note that the indicator function is always 1 because the analysis evaluates only the structural performance in the first cycle.

The Poisson process used to model earthquake occurrences implies that interarrival times between earthquakes are exponentially distributed; therefore,

$$
\lambda(t) = \frac{f(t)}{F(t)} = \frac{\mu e^{-\mu(t - T_0)}}{1 - (1 - e^{-\mu(T_0 - T_0)})} = \frac{\mu e^{-\mu(t - T_0)}}{e^{-\mu(T_0 - T_0)}} = \mu
$$

which reflects the memoryless property of the exponential distribution.

**Case 1: failure after first shock.** In case 1, it is assumed that damage does not accumulate with time and the structure is abandoned after the first failure. Then, the instantaneous failure probability can be computed according to Eq. (14),

$$
\mathcal{V}(t) = \int_{t_{s'}(t, s')/C_0}^{\infty} dG(y) \lambda(t) = \int_{t_{s'}(t, s')/C_0}^{\infty} dG(y) \mu \quad (20)
$$

If it is first assumed that the initial capacity is deterministic, i.e., $u = u_0$, then solution for case 1 (Fig. 4) becomes,

$$
\mathcal{V}(t) = 1 - \int_{0}^{u_0 - s'} dG(y)dy = 1 - G(u_0 - s')/\mu \quad (21)
$$

Note that in this case $V(t, s', n) = u_0 - s'$ and does not depend on $n$. For the case when the initial capacity is defined as a random variable it is necessary to uncondition Eq. (21) as follows:

$$
\mathcal{V}(t) = \int_{0}^{u_0 - s'} \mu [1 - G(u - s')] dH(u) \quad (22)
$$

where $dH(u) = h(u)du$ is the derivative of the distribution function of the initial capacity; i.e., lognormal with $\mu = 100$ (%) and $COV = 0.25$.

**Case 2: failure after successive shocks.** In case 2 (Fig. 4) damage accumulates with time and the lower limit of the integral in Eq. (20) becomes random and depends on the number of shocks, $n$. The accumulated damage is described by the sum of the damage caused by every shock. For the particular case where the damage caused by every shock is exponentially distributed, the accumulated damage after $n$ shocks follows an Erlang distribution,

$$
G^{(n)}(y, 0) = \frac{1}{(n - 1)!} \theta^n y^{n-1} \exp(-\theta y) \quad (23)
$$

where $y$ is the amount of damage and $\theta$ is the average damage size observed at every shock (i.e., damage rate). Then,

$$
\mathcal{V}(t, s', n) = \int_{0}^{u_0 - s'} (u_0 - s' - y)dG^{(n)}(y)dy \quad (24)
$$

Note that the term $u_0 - s'$ is the total capacity available to the system and $y$ is the capacity that has been taken from the system after $n$ shocks (damage). In other words, $V(t, s', n)$ defines the minimum damage size required to cause the system’s capacity to fall below $s'$ (Fig. 3). Similarly to case 1 (Fig. 4), the solution is conditioned on the number of shocks and on the initial capacity $u$. Then, by conditioning out these variables the final expression for the instantaneous failure rate in case 2 becomes:
\[ m(t) = \int_0^\infty \sum_{n=1}^\infty \mu(1 - G(V(t, u, s', n)))P(N_t = n)h_0(u)du \]  

\[ P(N_t = n) = \frac{(\mu t)^n \exp(-\mu t)}{n!} \]  

where the probability of having \( n \) shocks by time \( t \) with exponential interarrival times follows a Poisson distribution.

Comparison of cases 1 and 2. Comparing cases 1 and 2 (Fig. 5), it can be observed that case 2 (multiple shocks) leads to larger failure probabilities. This is expected since damage accumulates with time and the probability of exceeding the threshold increases with time. The solution presented in Eqs. (24) and (25) can be used to define time-dependent fragility curves that characterize various damage states (Fig. 6) by defining appropriate threshold values (i.e., \( s' \)). Since thresholds levels (i.e., \( s' \)) define damage states, instantaneous and accumulated probabilities of intervention are larger for minor damage states and smaller for severe damage states.

5. Progressive deterioration

Structures have many different deterioration mechanisms and the structural lifetime performance depends on how these mechanisms interact. The model of deterioration as a result of shocks only (Section 4) does not take into account the possibility that there exists a loss of capacity in between shocks; however, in real life it is common to observe both progressive deterioration and deterioration caused by shocks (i.e., rare events). Progressive degradation is usually a slow continuous time-dependent phenomenon, which is caused, for instance, by chloride ingress, corrosion, fatigue or biodeterioration [3,4]. Since there is little evidence of a strong correlation between progressive deterioration and damage as a result of shocks, in the proposed model independence will be assumed. Both deterministic and random models for deterioration are presented in this section.

5.1. Deterministic deterioration model

The first approach to modeling progressive deterioration is to describe it by a continuous deterministic function. For instance, in Fig. 7a both a linear and an exponential deterministic deterio-
tion models followed by a shock that causes the failure are shown. In addition, Fig. 7b presents the combined effect of various deterministic deterioration models combined with the effect of multiple shocks.

In the deterministic approach it is usually assumed that progressive degradation has a continuous positive rate \( r_p(t) > 0 \). This type of deterioration is commonly known in reliability as graceful deterioration [24]. Then, re-writing Eq. (3), the capacity of the structure by time \( t \) can be computed as:

\[
V(t) = u_{t-1} - \int_{Z_{t-1}}^{t} r_p(\tau) d\tau - Q(t)
\]

where the sub-index \( Z_t \) describes the cycle the system is in; \( Q(t) \) is the accumulated damage as defined in Eq. (5); and \( A_p(t, Z_{t-1}) = \int_{Z_{t-1}}^{t} r_p(\tau) d\tau \). By including a progressive deteriorating function, the integration limits in Eq. (9) need to be modified because progressive deterioration increases the loss of capacity reducing the shock size required to cause the failure (i.e., to exceed the threshold value). Then, similarly to Eq. (10), the integral limits in Eq. (9) can then be re-calculated as:

\[
V(t, s^*) = u_{t-1} - a^* - A_p(t, Z_{t-1}) - Q(t)
\]

Introducing the Stieltjes convolution of \( G(x) \) with itself,

\[
V(t, s^*, n) = \int_{0}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty} (u_0 - a^* - A_p(t, Z_{t-1}) - y) G(y) d\tau d\sigma d\omega
\]

where \( G(n)(y) \leq u_0 - a^* - A_p(t,0) \) or \( G(n)(y) \leq u_{t-1} - a^* - A_p(t, Z_{t-1}) \) depending upon the case.

Note that the only difference between Eqs. (30) and (13) is that there is a new term that accounts for progressive deterioration, which is deterministic and depends only on the rate \( r_p(t) \); i.e., \( A_p \). Also, in this model is possible to account for different deterministic deterioration functions defined through the appropriate selection of \( r_p(t) \). This deterministic continuous deterioration model can be applied in many cases where the uncertainty in graceful deterioration is not significant. Then, it can accommodate models commonly

![Fig. 6. A structure subject to multiple shocks: (a) instantaneous probability of reaching a given damage state by time \( t \); (b) accumulated probability of reaching a given damage state before or at time \( t \).](image6.png)

![Fig. 7. (a) Sample path for a single shock failure case and various deterministic degradation models; (b) sample path for the case of multiple shocks and various deterministic degradation models.](image7.png)
used in practice such as those proposed by Mori and Ellingwood [27] and Pandey [31].

5.2. Random progressive deterioration

The assumption that deterioration is a deterministic function works fine in many practical cases, however, if the uncertainty of the process needs to be included, a good approximation can be obtained by assuming that progressive deterioration is also a jump process in which the size of every jump is random; and jumps occur at fix time intervals. For instance, in structural systems this is equivalent to observed damage as a result of annual assessments (inspections) of the structural condition. Fig. 8a shows in light gray lines the deterministic separation of jumps in time and various sample paths. The combined effect of the jump model for progressive deterioration and the model of shocks (Section 4) is shown in Fig. 8b.

Define $B_i$ as the loss of capacity caused by a “progressive deterioration shock” $i$, and assume that these shocks are independent and identically distributed with $W(d) = PB < y)$. At a given time $t$ and in a cycle $L_t$, the accumulated damage $S(t)$ caused by progressive deterioration only can be computed as:

$$S(t) = \sum_{j=0}^{e(t)} B_i \cdot 1_{[z_i, z_{i+1}]}$$

where $e(t)$ is not a random variable but a known number.

Then, the structural capacity by time $t$ becomes:

$$V(t, a') = u_{L_t} - a' - Q(t) - S(t)$$

which results from subtracting, from the structural capacity available (i.e., $u_{L_t} - a'$), the deterioration caused by extreme overloads, i.e., shocks $Q(t)$, and the progressive deterioration in the form of small shocks occurring at fixed times, i.e., $S(t)$.

Eq. (13) can be extended to accommodate progressive deterioration; then, the integral limit required to compute the failure probability (Eq. (9)) becomes:

$$V(t, a', n) = \left[ \int_{0}^{a''} \int_{0}^{a'-y} (u_{L_t} - a' - y - d) dG^{(y)}(d) \right] dW^{(y)}(d) L_t = 1$$

where $dW^{(y)}(d)$ is the $e(t)$th convolution of the distribution $W$ with itself; and $dW^{(y)} = W^{(y)} dt$ is the density function of the sum of $e(t)$ shocks (describing progressive deterioration) up to time $t$.

Note that the term $d$ in Eq. (33) accounts for the loss of structural capacity as a result of progressive deterioration while $y$ takes into consideration the loss of capacity as a result of extreme overloads. In practice, because shocks that simulate progressive deterioration are evaluated at fixed times, it is possible to use this approach to simulate possible trends of the process (e.g., linear or exponential decay). Then, if the occurrence times are governed by any deterministic function $g(t)$, Eq. (32) becomes

$$V(t, a') = u_{L_t} - a' - Q(t) - S(t)$$

where $S(t)$ is the total damage caused by progressive deterioration as a result of $e(t)$ jumps that will occur by time $t$, according to function $g(t)$. The number of jumps $e(t)$ is computed by selecting the time between them, i.e., $\Delta t$, such that $(g(t_{i+1}) - g(t_i))/\Delta t = c$. For example, for $g(t) = x$, jumps are equally spaced, while for $g(t) = \exp(xt)$, $\Delta t$ gets smaller as time increases.

5.3. Illustrative example

Consider the same structure described in the previous example and assume that the system also suffers progressive deterioration with time.

5.3.1. Deterministic deterioration

Let’s initially consider only the model of graceful deterioration for the following two deterministic functions: $g_1(t) = t$ and $g_2(t) = \exp(xt)$, with $x > 0$. Then, if only the structural performance during the first cycle is considered, the integral limits for the deterministic progressive deterioration case can be defined by Eq. (30) as:

$$V(t, s', n)_{g_1(t)} = \int_{0}^{u_0 - s' - g_1(t)} (u_0 - s' - y - d) dG^{(y)}(d)$$

and for $g_2$: while for $g_2$, this limit becomes:

$$V(t, s', n)_{g_2(t)} = \int_{0}^{u_0 - s' - \exp(xt)} (u_0 - s' - y - d) dG^{(y)}(d)$$

Fig. 8. (a) Sample path of random progressive deterioration for linear and exponentially time-spaced shocks; (b) shocks models used to simulate both extreme events and progressive deterioration.
5.3.2. Random progressive deterioration

If randomness is introduced in both deterioration models by describing them as a jump processes with fix interarrival times, the integral limits defined in Eq. (33) become,

\[ V(t, \tau, n)(\beta) = \int_0^{\tau - t} \left[ \int_0^{\tau - s - d} (u_0 - s - d - y) dG^\kappa(y) \right] dW^{\nu_0(\beta)}(d_1) \]

(37)

\[ V(t, \tau, n)(\beta) = \int_0^{\tau - t} \left[ \int_0^{\tau - s - d} (u_0 - s - d - y) dG^\kappa(y) \right] dW^{\nu_1(\beta)}(d_2) \]

(38)

In both random progressive deterioration models it was assumed that the damage of jumps is exponentially distributed and jumps are iid. Note that since jumps are exponentially distributed, \( dW^{\nu_0(\beta)}(d) \) will follow an Earlang Distribution.

5.3.3. Results and discussion

In this example, the rate of jump sizes was assumed to be \( \vartheta = 0.75 \) and the intervention threshold (i.e., limit state) was defined by \( s' = 0.25 \). The integral limits for all deterministic and random cases are shown in Fig. 9 (Eqs. (35)–(38)). It can be observed that by introducing randomness, the integral limits change reducing the structural capacity available with respect to the deterministic case.

Fig. 10 presents the instant intervention rate for various performance models. It can be observed that when multiple shocks are taken into account, the instantaneous failure rate increases and grows smoother than when a single shock model is used. The accumulated probabilities of intervention by time \( t \) are shown in Fig. 11 for the same cases shown in Fig. 10. As expected, the accumulated probability of intervention increases when progressive deterioration is included. Also random deterioration models lead to larger intervention probabilities than the deterministic models.

6. Total system failure

As discussed in previous sections, the system may fail as a result of shocks due to random extreme overloads, progressive deterioration or a combination of both. Although the loss of capacity is a consequence of both processes, it is assumed that at the time of failure they will not occur simultaneously and that the actual failure is caused by one mechanism only. Then, the instantaneous intervention rate, can be written as:

\[ v(t) = \left[ \left( \int_{y_{j+1}(\lambda)}^{y_{j}(\lambda)} dG(y) \right) \lambda(t) + \delta(t) \int_{y_j(\lambda)}^{\infty} dW(y) \right] \text{Failure} \]

\[ \left( \int_{y_j(\lambda)}^{\infty} dW(y) \right) \lambda(t) + \delta(t) \int_{y_j(\lambda)}^{\infty} dW(y) \right] \text{Prev. Maint.} \]

(39)

Note that in Eq. (39) the second term in the expression within brackets accounts for progressive deterioration. The term \( \delta(t) \) is indicator function that takes the value of 1 at times of “scheduled” fictitious shocks (describing the progressive deterioration) and 0 otherwise. This means that if a fictitious shock discribing progressive deterioration is not programed, failures can only occur as a result of shocks caused by extreme overloads.
7. Regenerative characteristics of the process

If a structure is systematically reconstructed (after failure or intervention), its performance with time can be modeled as a renewal process. In renewal processes, if the origin of the analysis coincides with the initiation of a cycle, the distribution function to the first renewal and the distribution of any other cycle are equal. These types of processes are called ordinary renewal processes. On the other hand, if the renewal process is said to be delayed if the origin does not coincide with the initiation of a cycle; in this case, the first renewal has a different distribution than the subsequent cycles.

For systematically reconstructed structures, the cycle within which the structure is at the time of evaluation becomes important in the assessment. However, it can be conditioned out by finding a limiting solution assuming that the effects of the origin vanish as \( t \to \infty \); i.e., the process has been running for a long time. Note that in this case the delayed and ordinary processes converge asymptotically although the transient behavior is different. Then, for structures that have been in operation for a long time, the asymptotic solution for the instantaneous failure probability can be expressed as [6]:

\[
\lim_{t \to \infty} \mathbb{V}(t) = \frac{1}{E[L]} \sum_{n=0}^{\infty} \left( \int_{\mathbb{V}_{[a,a')}} \mathbb{X}(t) dG(y) \right) \mathbb{X}(t) P[N_i = n]
\]

where \( E[L] \) is the expected value of the length of a cycle. The length of one cycle is the expected time between interventions given that repair or reconstruction times are not significant with respect to the total life-cycle.

8. Discussion and conclusions

The paper presents a stochastic model that can be used to characterize the performance of structures that deteriorate as a consequence of both sudden events i.e., shocks (e.g., earthquakes, terrorist attacks, accidents) and progressive degradation (biodegradation, sulfur attack, corrosion, fatigue). The model describes the system’s performance in terms of its capacity, which in practice would describe the structural response such as, for instance, inter-story drift or maximum displacement, resistance or safety availability. It takes into account the accumulation of damage as a consequence of both successive shocks and progressive deterioration. Shocks describe structural damage and are modeled as independent events that occur randomly in time and whose size is also random. Progressive deterioration was first modeled as a deterministic continuous function. Afterwards, randomness was included by modeling it as successive small shocks occurring according to a deterministic function. In addition, to provide the means to compute the system’s failure probability, the model provides the basis for modeling and defining preventive maintenance strategies. For instance, it provides flexibility to define different damage limit states. Also, it can be used to implement intervention policies based on operation requirements defined by capacity thresholds (i.e., \( s^* \) and \( k^* \)).

The illustrative examples presented and discussed stress the importance of taking into account the accumulation of damage caused by both shocks and progressive deterioration on the probability of failure. Results show that the probability of failure can be significantly higher if these progressive failure mechanisms are considered. Therefore, most existing models overestimate the structural reliability.

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